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# LINKAGES

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A HISTORICAL study of linkages starts with the work of James Watt, although the pantagraph and lazy-tongs, both familiar cases of simple linkworks, were known at least as early as the seventeenth century. In his improvements upon the steam-engine the Scottish engineer found it necessary to guide the piston-rod in a straight line, and yet to communicate its motion to the circular movement of the working-beam. A diagram of his invention of the purpose is shown in Figure 1; the combination of three links provided an approximate rectilinear motion whose "sweet simplicity" was astonishing when contrasted to the double chains or racks and sectors which it replaced. In a letter to his son upon the subject he concludes, "Though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other mechanical invention I have ever made."[1] Such praise from Watt is praise indeed; his interest in link motions was a lasting one, and he spent the later years of his long and splendid life in perfecting a machine for copying sculptures, a sort of pantagraph in space, which was to act with all the "delicate smoothness" of the parallel motion.

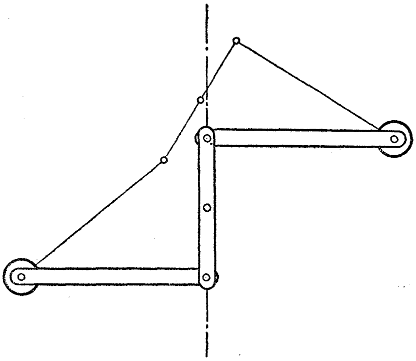

FIG. 1. WATT'S PARALLEL MOTION.

It was in the "latter end of 1783" that Watt devised the application to the steam-engine, but, in spite of the fact that it was almost universally used, very few adaptations or other combinations of links for the conversion of motion were tried for many years. Examples of Watt's parallelogram were given by engineering writers, such as the famous and influential Frenchman, Prony,[2] but no comprehensive investigation was made until the middle of the nineteenth century, when the impulse came from the University of St. Petersburg.

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[1] See D. H. Leavens, *Am. Math. Monthly,* Vol. 22, 1915, p. 331. He quotes from J. P. Muirhead, "Life of James Watt," New York, Appleton, 1859, p. 242. The reference to the second edition (London, 1859) is p. 288.

In 1847 Pafnouty Lvovitch Tchebychef obtained a position at the University of St. Petersburg, and became able to indulge somewhat an interest he had pursued from a child,[3] the construction of mechanisms and models of his own invention. In the vacation of 1852 he took a trip through Western Europe to visit factories, to see different types of mechanisms and, most of all, to study Watt's parallel motion; towards the close of the summer he crossed the Channel to pay his respects to Cayley and Sylvester, and to Gregory the engineer. In London, Tchebychef hunted up some of Watt's original machines, preserved in the Royal Polytechnic Institute, and satisfied his inquiry by measuring the lengths of all accessible parts, and studying the details of the various arrangements.

On his return to Russia Tchebychef devoted a large part of his time and enthusiasm to the question of providing an analytical method for handling such motions as that of Watt's parallelogram. Recognizing the advantages of a geometrical insight into such motions, he nevertheless regretted the difficulty of a geometrical grasp of the subject, and for himself preferred methods of analysis. His own methods were remarkably clever, if somewhat prolix; and they enabled him to devise new motions whose deviations from a straight line were given in terms of arbitrary constants of the linkwork, and which were so inconsiderable as to be within the limits of mechanical accuracy.

But although these approximations may have been satisfactory from the point of view of mechanical motion, they were by no means a sufficient solution of the problem of drawing an exactly straight line. To accomplish this Tchebychef's analysis was unavailing. So far three links had been employed in every model, and the curve traced by points upon the traversing bar was beginning to be studied, a sextic of an analytically unattractive appearance. It was hardly likely that a five-bar linkwork, or a more complicated model, would produce a more simple result. Indeed, for more than three links Tchebychef's analysis grew so involved that a linear solution was extremely improbable; and when he became certain that no three-bar motion could trace a straight line, he is said to have considered it impossible to produce rectilinear motion by any linkwork.[4]

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[2] "Nouvelle Architecture Hydraulique," 1796, t. 2, p. 123. See Muirhead, second edition, p. 261. The terms "parallel motion" and "parallelogram," though not at all descriptive, are used throughout the early literature, and may therefore be retained here.

[3] A biographical sketch is given in Oeuvres de P. L. Tchebychef, St. Petersburg, t. 2 (1907), pp. i–vi, and his account of the trip, pp. vii–xix. (Cf. Leavens, *loc. cit.*)

But in this result the eminent Russian scientist decided "with less than his usual success in overcoming difficulties insuperable to the rest of the world," and it seems as though the spirit of science was merely waiting for this confession of inability to bring to men's attention the simplicity of the true solution. The first rectilinear motion by linkages was invented by Lieutenant Peaucellier, a young French officer of engineers detailed for the moment, and possibly with consequent leisure, to staff duty; a subsequent discovery was made by L. Lipkin, a freshman at the University of St. Petersburg, who was studying mechanics under Professor Tchebychef. Peaucellier's discovery was made about 1864, Lipkin's in 1870; and an interesting fact is that, although the two were independent, their methods and linkworks were precisely the same.

The linkwork of Peaucellier and Lipkin jumps from three to seven links. There are first of all two long links of equal length, both pivoted to the same fixed point; their other extremities are pivoted to opposite angles of a rhombus composed of four equal shorter links, as shown in Fig. 2. The portion of the linkage so far described is the essential part, and is called Peaucellier cell; by means of a seventh and extra link the cell is made to move so that the free end of the rhombus describes an accurately straight line.

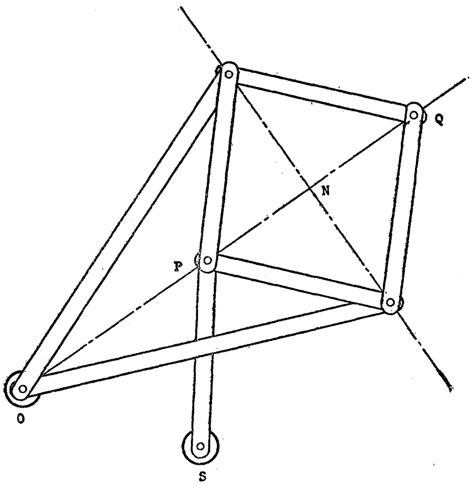

FIG. 2.

The reasoning to prove this is very simple and depends upon a fundamental property of geometrical inversion. It is evident that whatever shape and position the linkage assumes, the points $O$, $P$, $Q$ will always be on a line; and if $N$ be the intersection of the diagonals of the rhombus,

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[4] See J. J. Sylvester, "On Recent Discoveries in Mechanical Conversion of Motion," *Proc. Roy. Inst.*, Vol. VII., pp. 181 and 183, footnotes.

$$OA^2 = ON^2 + AN^2$$

$$AQ^2 = QN^2 + AN^2$$

so that

$$\begin{aligned} OA^2 - AQ^2 &= ON^2 - QN^2 \\ &= (ON - QN)(ON + QN) \\ &= OP \cdot OQ \end{aligned}$$

Hence if $O$ is pivoted to a fixed base, since $OA$ and $AQ$ are the constant and arbitrary lengths of the links, the product

$$OP \cdot OQ = \text{constant}$$

so that whatever the shape of the cell, $P$ and $Q$ are inverse points with respect to a circle whose center is at $O$; and whatever curve $P$ is made to trace, its inverse curve will be traced by $Q$.

So stated, the remainder of the problem is not difficult; for if we want $Q$ to trace a line, an easy way will be to make $P$ trace a circle passing through $O$, since the inverse of any circle through the center of inversion is a line. Hence the extra link, pivoted to $P$ and to a fixed point $S$ whose distances from $O$ and $P$ are equal. Then any motion of the linkwork will cause $P$ to move on a circle through $O$, and $Q$ will move along a line.

This pretty solution of the problem of rectilinear motion was published by Peaucellier in 1864; unfortunately not in a complete form, but merely as an announcement in the *Nouvelles Annales* for that year; and coming as it did from an unrecognized person in an unorthodox way it was "successfully concealed under a wrong entry." At any rate, the little announcement was quite overlooked by those interested, until six years later when the work was duplicated by the Russian student. Then Professor Tchebychef's admiration for the work of his pupil, although it confuted his own analysis, secured for Lipkin a handsome reward from the Russian Government. Tchebychef's enthusiasm communicated the results to his friends in France and England, and the stock in linkages rose. The hitherto unrecognized Peaucellier, now a colonel in command at Toul, was exhumed from his dugouts and awarded the famous "Prix Montyon" of the Institute of France; and it appears that the two young men who are here linked together for their independent yet similar abilities, were similarly sated by success, for neither Peaucellier nor Lipkin is again heard of in the history of science.

But if the younger men were resting quietly upon their laurels, their work was seized upon with complimentary rapidity by men of more maturity and standing; and at this juncture the London school of mathematics became interested in the results which had hitherto been confined to engineers, and mostly obtained upon the continent. In 1869 Samuel Roberts suggested the connection between geometry and linkages in a paper read before the society "On the Mechanical Description" of certain curves.[5]

Mr. Roberts chose as his text the saying of Newton, "At æquatio non est, sed descriptio, quæ curvam Geometricam efficit," and considered that "in the present state of the theory of quartic and cubic curves, it is very desirable that we should be able to draw them continuosuly." He cited the classic examples of Nicomedes' Trammel for drawing the conchoid, Newton's method for the cissoid, and Pascal's for the limaçon. By generalizing these methods some interesting results were obtained; and among the more important results was the fact that Cayley became interested in the drawing of curves, and began a discussion of the three-bar curve, or the curve traced by any three-bar linkwork such as that of Watt.

The time was ripe for the introduction of Peaucellier's motion into England, and perhaps the dramatic way in which it was rediscovered by Lipkin had something to do with the interest it aroused. Consider the effect of a sincere statement by an eminent man that it was impossible to convert circular into rectilinear motion, a statement, by the way, which was confused by many with the squaring of the circle; and when this noted man had been for some time buried in a mass of computation to prove his point, a timid freshman raps upon the door, presenting to his eyes the thing itself, and not the proof of its impossibility. It was very likely this paradoxical entrance that pleased Sylvester. At any rate, Sylvester took up the invention with enthusiasm, enlisted Professor Henrici's services to procure him models, and at the annual general meeting of the London Mathematical Society in 1873 he gave a talk which was, by the secretary's account, "warmly applauded."[6] Following that, he made a well-known address[7] at the Royal Institute to a more general audience; and the enthusiasm with which that facile scientist spoke upon the subject to every one he met played a great part in its popularity, and the practical application which he foretold lent to it a specious importance.

The interest which Sylvester himself took in linkages is well attested by the address mentioned above; taken down by the secretary, it was probably given to the speaker for annotation; and when returned to the secretary for publication it had considerably more than five times as much in the footnotes as in the text, with a postscript superadded. In this postscript were included his "kinematical paradox" for representing a constant as a kinematical function of the independent variable, and several schemes for representing crystallographic and atomic groupings by means of linkages. "It would be difficult to quote any other discovery," he writes, "which opens out such vast and varied horizons as this of Peaucellier—in one direction, as has been shown, descending to the wants of the workshop, the simplification of the steam-engine, the revolutionizing of the millwright's trade, the amelioration of garden-pumps, and other domestic convenience (the sun of science glorifies all it shines upon), and in the other soaring to the sublimest heights of the most advanced doctrines of modern analysis, lending aid to, and throwing light from a totally unsuspected quarter on the researches of such men as Abel, Rieman, Clebsch, Grassman, and Cayley. Its head towers above the clouds, while its feet plunge into the bowels of the earth."[8]

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[5] S. Roberts, *Proc. Lond. Math. Soc.*, Vol. 2, 1869, p. 125.

[6] *Proc. Lond. Math. Soc.*, Vol. 5, p. 4.

[7] J. J. Sylvester, "On Recent Discoveries in Mechanical Conversion of Motion," *Proc. Roy. Inst.*, Vol. 7, p. 179; or "Collected Works," Vol. 3, p. 7.

Although Sylvester's words read rather like an advertisement than sober science, his enthusiasm was adopted by many round him and the models were admired by many more. He showed a model of Peaucellier's cell to Sir William Thomson, and according to Sylvester the canny Scot "nursed it as if it had been his own child, and when a motion was made to relieve him of it, replied, 'No! I have not had nearly enough of it—it is the most beautiful thing I have ever seen in my life.'"[9] Considering the extraordinary conversions worked with the model, Sylvester considered that "it would not be unsuitable to write in letters of gold on the board attached to it which gives support to the two frail centers, the famous motto of Constantine —'In hoc signo vinces.'"

With Sylvester pushing the subject from one side and Cayley from the other, the interest in linkages could not fail to be keen, and many other men were drawn to the subject; A. B. Kempe and Harry Hart in England, Lemoine and Brocard on the continent, Woolsey Johnson in America, and an even more notable trio, Darboux, Clifford and G. H. Darwin. Within ten years close to one hundred and fifty papers had been published in recognized journals;[10] and if a curve is plotted as in Fig. 3 to show how many papers were published every year, as an indication of the interest, such a curve would show a remarkable rise during 1874, and its culmination reached in 1875 with thirty-six articles published in that year. The rapid rise of the curve is no more remarkable than its sudden fall; and the shape in general is typical of an explosive epidemic, due to powerful causes, and rapidly running its course to a conclusion.

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[8] Sylvester, *loc. cit.*, p. 195. His spelling, Rieman and Grassman, is preserved.

[9] Sylvester, *loc. cit.*, p. 183.

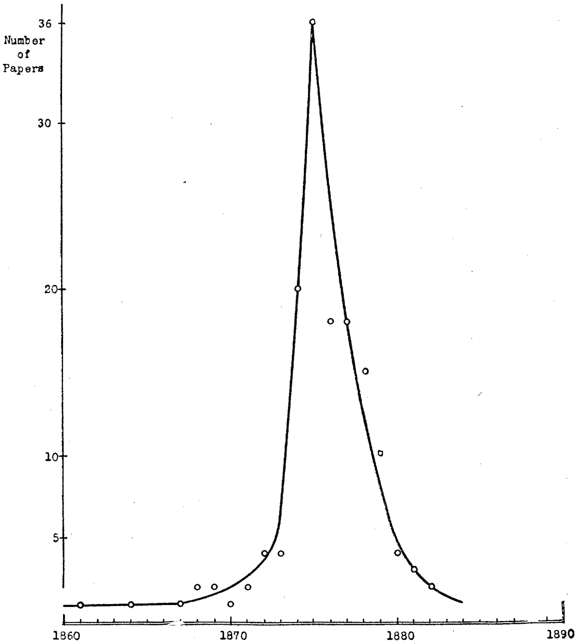

Fig. 3.

Peaucellier's motion was introduced by Sylvester into England in 1873, and many up to date mechanics seized upon the principle. Some circular steps outside St. Paul's Cathedral were so worn as to require repair, and the surveyor used a cell, adjusted to circular instead of rectilinear motion, to cut the templets; and though the radius of the steps was about forty feet, the cell operated "to the great comfort and delectation of his clerk" with a link some six feet long.[11] It was, by the way, this same surveyor whose pump was "ameliorated" by Peaucellier's principle. And a little later came the classic example which all students of linkages quote, the application of the motion to the air engines which ventilate the House of Parliament.

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[10] V. Liguine, *Bull. Sci. Math.*, 2d series, Vol. 7, 1883, p. 145, gives a bibliography of 151 papers up to 1882.

[11] Sylvester, *Proc. Roy. Inst.*, *loc. cit.*, p. 182.

In August of 1874 Harry Hart of Woolwich Academy disproved Tchebychef's last entrenchment[12] by showing that Peaucellier's cell of six links could be replaced by a new linkage of only four bars. Hart's linkage was obtained by crossing the links of an ordinary jointed parallelogram, as in Fig. 4, forming what is called a contra-parallelogram. Then the four midpoints[13] have exactly the properties of the points of a Peaucellier cell to which two long links have been added for symmetry; and it follows that a five-bar rectilinear motion may be produced from Hart's contra-parallelogram by the addition of an extra link.[14]

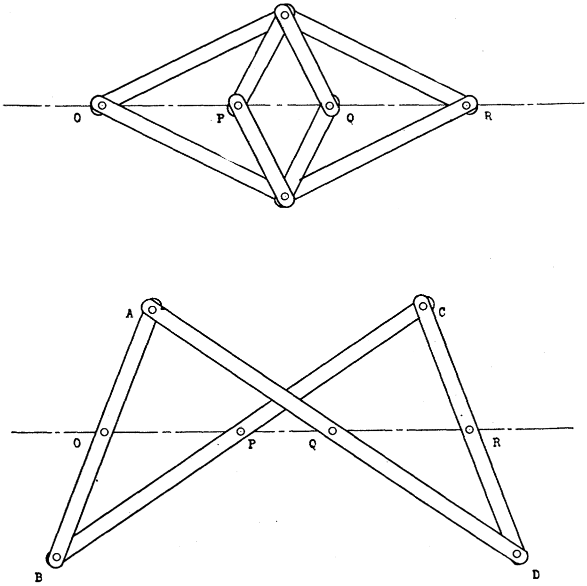

Fig. 4. Hart's Contra-parallelogram.

So far linkages had been studied from an experimental standpoint, as arrangements of a certain number of rods and pivots; but about this time Samuel Roberts suggested that instead of considering the motion of jointed rods it would be better to consider the motion of the planes associated with those rods, for the order of the path-curves would not thereby be increased. Sylvester then substituted the more general idea of the relative, as distinguished from the absolute, motion of plane upon plane; a conception which is fruitful in indicating that link-motion may be reduced to the rolling of centrodes.[15]

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[12] Tchebychef stated the impossibility of five-bar rectilinear motion, even after Peaucellier's discovery was known. See Sylvester, *loc. cit.*, p. 181.

[13] Or any points whose ratio $AO : OB$ is constant.

[14] A different and pretty five-bar linkwork for drawing a straight line was given by M. Raoul Bricard in 1895 [*Comptes Rendus*, 120, p. 69], but as not related to the general story of the epidemic it is not included here.

In studying the ordinary three-bar motion it had occurred to Kempe[16] to consider what happened when the traversing bar and one of the radial bars had changed places, and the conclusion reached without difficulty was that the order in which the bars are chosen does not affect the shape of the path-curve. If, in Fig. 5, *OC, CB, BS* be the original three bars, and *P* a point on the traversing bar tracing a certain curve; then if the two bars *OA*, equal to *CB*, and *AB*, equal to *OC*, are added, and *Q* is on *AB* such that

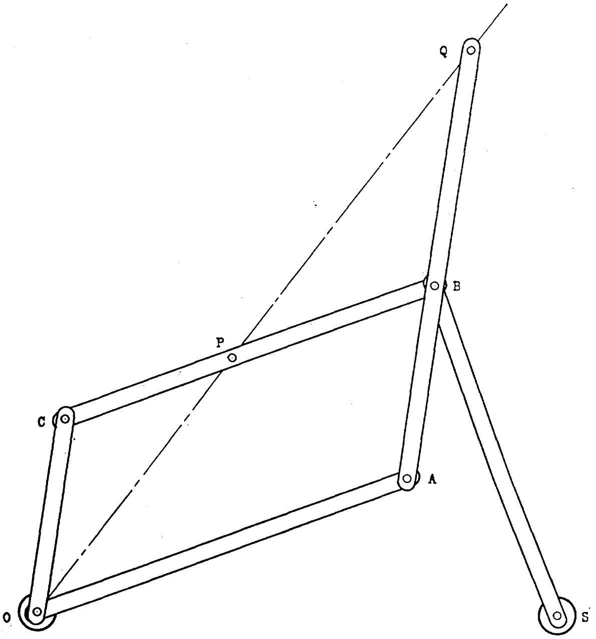

FIG. 5. PANTAGRAPH.

$$QO : PO = AO : CO,$$

*Q* will trace a curve similar in shape and position to that traced by *P*, but enlarged in the fixed ratio. And if the bars *OC* and *CB* are taken away there remains a three-bar linkwork which is the same as the old except that the radial link and the traversing link are interchanged. This was remarked by Sylvester as a most acute and admirable theorem; but is also, as Cayley observed when Sylvester consulted him, a self-evident deduction from the principle of the ordinary pantagraph.

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[15] See Sylvester, "History of the Plagiograph," *Nature*, Vol. 12, 1875, p. 214, footnote.

[16] A. B. Kempe, "How to Draw a Straight Line," London, Macmillan, 1877, p. 20; Sylvester, *loc. cit.*, p. 215.

The result obtained by Kempe, when communicated to Sylvester, was immediately seen by the latter to be extensible to the case of three-piece motion, where the bars are replaced by their planes; and Sylvester's skew pantagraph or "plagiograph" is obtained by making $P$ and $Q$ similarly situated in the planes of $CB$ and $AB$, as by making the triangles $CPB$ and $AQB$ similar, in Fig. 6. Then $P$ and $Q$ trace curves which are similar, but turned through a fixed angle.

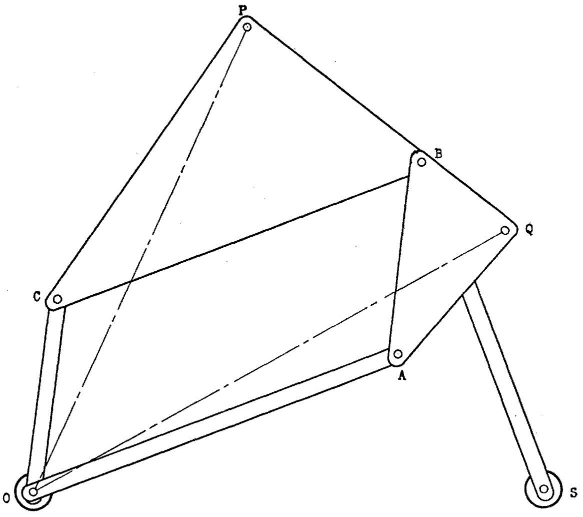

FIG. 6. PLAGIOGRAPH.

It is a natural conclusion that the principle of the plagiograph might be applicable to Hart's contra-parallelogram, and both Kempe and Sylvester, the one "by the free play of his vivacious geometrical imagination," the other "by the sure and fatal march of algebraical analysis," found that if a chain of similar triangles be attached to the four links, the free vertices will form a parallelogram whose angles are invariable and whose area is constant. For instance, to use the same lettering in Fig. 7 as in Fig. 4, if $O$, $P$, $Q$, $R$ are the vertices of isosceles right triangles fixed to the bars $AB$, $BC$, $AD$, $CD$, then $OPQR$ is always a rectangle of constant area. The linkage is, by Sylvester's name, a "quadruplane."

The quadruplane affords much light upon three-bar motion, which results when any one plane is held fixed. If, for example, $AB$ is fixed, then $I$ will be the instantaneous center, and

$$AI + BI = BI + CI = BC, \text{ a constant,}$$

so that the point $I$ describes upon the fixed plane an ellipse of which $A$, $B$ are the foci and $BC$ the length of the major axis. In a similar manner

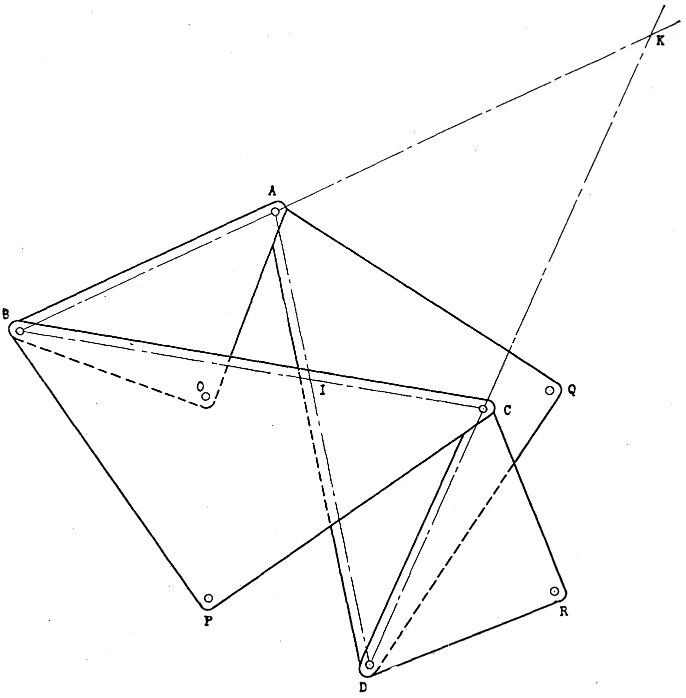

FIG. 7. QUADRUPLANE.

$$CI + DI = DA$$

and the locus of $I$ on the moving or traversing plane $CD$ is an equal and similar ellipse. Hence the relative motion of the two opposite planes $AB$, $CD$, is that produced by the rolling of two equal and similar ellipses.

The two planes $BC$, $AD$ have their instantaneous center at $K$; the trace of $K$ upon the plane $BC$ is given by

$$BC - CN = BN - AN = BA,$$

an hyperbola, and its trace upon the plane $AD$ is an equal and similar hyperbola. It follows that the free motion of the quadruplane, or the relative motion of four planes so linked, reduces to the double rolling of two ellipses and two hyperbolas.

This conception of three-bar curves being produced by the rolling of two equal conics shows that the particular quartics produced are related to the inverses of conics, and simplifies the discussion considerably; and in broad terms this represents the state of knowledge of three-bar motion through the middle of

1875, when Sylvester had published an article in *Nature*. In November of that year Samuel Roberts made an important observation in the theory of three-bar motion proper, or the motion of three links about two fixed pivots. Noticing that the pivoted points turn up as singular foci in the discussion of the sextic path-curves, and that there are three foci to the sextic, he suggested that the two pivots might be placed at any two of the three foci, and by means of links of suitable lengths the same locus would be obtained. In other words, any path described by a point moving with three-bar motion may also be described in two other ways by three-bar motion.[17]

Whatever the merits of Mr. Roberts per se—and this observation was singularly acute—he seems to have had the important ability to interest Professor Cayley; and once again Cayley was led to an investigation of three-bar curves by a consideration of Roberts's results. In March of 1876 he put the above theorem into the following elegant form.[18] Take any triangle *ABC* and through any point *O* within it draw lines *KF, EH, GD* parallel to the sides. Let the triangles *HKO, GOF, ODE* be supposed rigid and jointed together at *O*, and let the other lines in the figure represent bars forming three jointed parallelograms, as shown in Fig. 8. Then however the system is moved about in its plane the triangle *ABC* will always be of the same shape; and further, that starting from any given position of the three triangles, the linkage may be so moved as not to alter the triangle *ABC* in magnitude; so that when the three points *A, B, C* are fixed in any other than their maximum position, the point *O* will still remain movable. In so moving, *O* will describe a path which is due at the same time to three different three-bar motions.

These pretty results were proved by Cayley with a somewhat forbidding nomenclature, and Clifford, who usually sat in the back of the room and said little at the meetings of the society, gives a very much simpler discussion in his "Kinematic."[19] But the two papers which Cayley read at this time contain much information on the three-bar curve; and, though they by no means say the last word upon the subject, no further word was said for quite a time. For reference once more to Fig. 3 will show that in 1876 the interest in linkages had passed its climacteric, and that the number of papers published and

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[17] S. Roberts, *Proc. Lond. Math. Soc.*, Vol. 7, p. 17.

[18] Cayley, *Proc. Lond. Math. Soc.*, Vol. 7, 1876, p. 136; cf. Clifford, "Kinematic," p. 149.

[19] Clifford, *loc. cit.* Curiously enough, the same simplification was published as original by Hart [*Camb. Mess. Math.*, Vol. 12, 1882, p. 32], while Clifford's "Kinematic," surely a widely read book, came out in 1878.

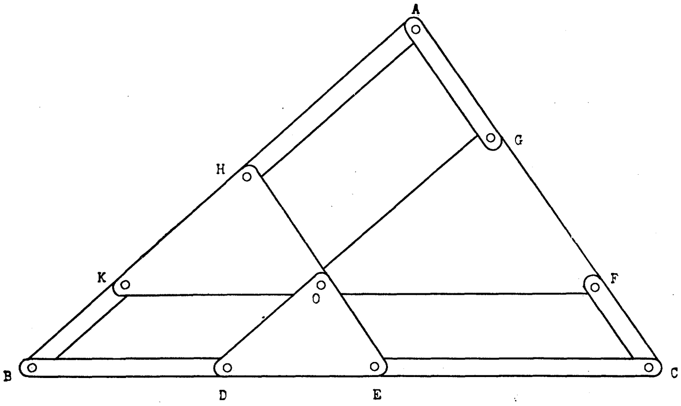

FIG. 8. TRIPLE GENERATION OF THREE-BAR MOTION.

new results obtained was rapidly diminishing. One stopping point upon the downward course occurred when Kempe proved the rather remarkable theorem that any algebraic curve whatever can be described by a linkwork, and his analysis is as beautiful as his result is elegant.[20] But after that the enthusiasm fell as fast as it had risen, and the epidemic, as far as history is concerned, was over.

Sir William Osler quotes Sidney Smith as saying, it is not the man who first says a thing, but it is he who says it so long, so loudly, and so clearly that he compels men to hear him—it is to him that the credit belongs; and so far as this singular epidemic of linkages is concerned, the credit for its existence belongs to Sylvester. It was through his efforts that other great minds turned to the subject; and indeed, the subject is made more interesting than many other offshoots of science only by the notable character of its contributors. And it may be more than a coincidence that the epidemic failed just at the time when Sylvester became involved in building up a new department in another land. The students at the Johns Hopkins University were interested in other lines, and though there are several papers on linkages in the *American Journal of Mathematics*, there are none of great importance. In England, Cayley and Kempe were diverted to map-coloring and the study of groups; of the younger men, Darwin found another line more suited to his taste, and the incomparable Clifford died before we had a chance to learn what he might have done. With the loss of the contributions of these leaders the first chapter of a study of linkages may close.

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[20] Kempe, *Proc. Lond. Math. Soc.*, Vol. 7, 1876, p. 213.